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On the evolution of stock vectors in a deterministic integer-valued Markov system

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Abstract

For multistate systems, studies on the evolution of the stock vector have already been done under various hypothesis on the model. For example for a regular Markov chain it is known that the system evolves towards a unique limiting vector, that is a fixed point of the transition matrix and that is independent on the initial stock vector.

In this paper, there are examined, under deterministic assumptions, constant size systems in which the integer-valued stock vectors are generated by a regular Markov chain. For these models, properties of the evolution of the integer-valued state sizes are proved and the limiting stock vector is examined. Sufficient conditions on the transition matrix are formulated to have, for two initial integer-valued stock vectors with l_p-distance equal to 1, at each step of the trajectory of evolution integer-valued vectors of which the corresponding coordinates differ at most with one.

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0. Introduction

Markov chains are a useful modelling tool in various fields, among others the field of manpower planning [1,2] and demography [3]. Under deterministic assumptions, in population studies one

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of the goals is to gain insight into the evolution of the number of people for each of the states, i.e. the evolution of the stock. For a system with k states, initial stock vector $n = (n_1, \ldots, n_k)$ and total size $N = \sum_i n_i$ and that is modelled by a Markov chain with transition matrix A, the vector $n \cdot A^{\mathrm{t}}$ is an element of the set

$$C_k = \left\{ u \in \mathbb{R}^k \,\middle|\, \sum_i u_i = N,\ u_i \geqslant 0 \right\}$$

and represents the stock vector after t periods of time.

In manpower planning, for example, each state of the Markov chain is corresponding with a personnel category that is homogeneous with respect to the transition probabilities (as there are promotion probabilities). In a deterministic approach, the implementation of the promotion- and recruitment strategy, reflected by the transition matrix of the Markov chain, results in stock vectors of which the components refer to the number of personnel in the different categories. Therefore, in discussing the evolution (according to the strategy) of the number of employees in each of the categories, at each step an integer-valued stock vector has to be considered. To deal with these restrictions, an integer-valued model (as in [4]) can be considered in which the integer-valued stock vectors are generated by a Markov chain with transition matrix A: having an integer-valued stock vector $n(t)$ at time t, the stock vector at time $t + 1$ is an integer-valued vector corresponding with $n(t).A$. In what follows, to this model there is referred to as the integer-valued model with transition matrix A.

In some deterministic discussions in population studies (e.g. [1, p. 109]), the evolution of the number of individuals in the states is described based on the properties of the vectors $n, n \cdot A$, $n \cdot A^2, \ldots$ and/or $\lim_{t\to+\infty} n \cdot A^{\mathrm{t}}$ itself, neglecting the condition on the stock vector to be integer-valued. Nevertheless it is worth to focus on this problem, since in [5] it is illustrated by an example that there can be a substantial discrepancy between the evolution of the vectors $n.A^{\mathrm{t}}$ on the one hand and the evolution of the stock vectors in the integer-valued model on the other hand.

In other discussions (e.g. [1, p. 112]) one has dealt with the problem of integer-valued stock vectors, by associating with each component of the computed vector $n \cdot A^{\mathrm{t}}$ the rounded integer. That is an approach not taking into account the restriction that a system modelled by a Markov chain has a total size that is constant and the fact that the stock vectors are assumed to be elements of the set C_k.

In previous work [4], for a starting vector n there is formalized which integer-valued vectors can be associated with the computed result $n \cdot A$. And integer-valued models based on a regular Markov chain with transition matrix $P + w'r$ are introduced, for constant size personnel systems characterised by a promotion matrix P, a wastage vector w (w' refers to the transpose of the row vector w) and a recruitment vector r. Moreover, for these models, properties of calculated predictions of state sizes and the associated integer-valued vectors were discussed.

For a regular Markov chain with transition matrix A, it is known from the Perron–Frobenius Theorem [6, p. 1] that independently on the initial stock vector, the system (with total size N) will evolve necessarily towards n^*, the unique fixed point of the matrix A in the set C_k. Fixed-Point Theorems are proved under several conditions on the function. A review of important Fixed-Point Theorems is given in [7, Chapter 3] and in [8].

In [5], the set of fixed points is examined and described analytically, for integer-valued constant size models based on a regular Markov chain with transition matrix A. In the context of an integer-valued model, an integer-valued vector n is said to be a fixed point if n is an integer-valued vector associated with the calculated vector $n \cdot A$. From previous results [5], it is known that under the condition of integer-valued stock vectors, depending on the matrix A, the set F of fixed points may either contain no, exactly one or more than one integer-valued stock vector. This result is in contrast with the uniqueness of the fixed point for a regular Markov chain.

In this paper the evolution of the subsequent integer-valued stock vectors is examined. For an integer-valued model, in contrast with the underlying regular Markov chain, not all initial stock vectors converge to a limiting stock vector. But in the situation that the subsequent integer-valued stock vectors do converge, the limiting stock vector is necessarily an element of the set F of fixed points. Besides, in case there is more than one integer-valued fixed point, the limiting vector can vary depending on the starting stock vector.

The consequence of all this is that modelling a phenomenon by an integer-valued model based on a regular Markov chain, instead of by the regular Markov chain itself, give rise to supplementary aspects in the discussion of the evolution towards a limiting stock vector. For example for a manpower system modelled by a regular Markov chain, in practice the unique limiting stock vector n^* is treated as a vector characterizing the evolution of the number of members in the personnel categories. And more in particular, in a personnel system controlled by recruitment and/or promotion, the recruitment vector r and/or the promotion matrix P will be determined so that the Markov chain with transition matrix $A = P + w'r$ evolves to a desired limiting personnel stock vector n^* [1, p. 264]. One should be aware of the fact that implementing that particular promotion and recruitment strategy, resulting in an integer-valued model with transition matrix $A = P + w'r$, does not guarantee the preferable evolution towards n^*. Since in [5] there is pointed out that for some transition matrices and for some starting stock vectors, there is a significant discrepancy between the actual limiting situation for the integer-valued model and the limiting stock vector n^* of the Markov chain.

For these reasons, it is worth to have a better understanding of the properties of the evolution of the subsequent integer-valued stock vectors over the different states towards a limiting stock vector, depending on the starting stock vector. An interesting question that can be stated is whether similar starting stock vectors will result or not in similar trajectories of evolution and in similar limiting situations. In this paper for the integer-valued model with transition matrix A, there are formulated conditions on A under which two initial integer-valued stock vectors that are 'neighbours' (i.e. vectors with l_∞-distance equal to 1) do result, at each intermediate step of the trajectory of evolution as well as in the limiting situation, in integer-valued vectors that are still 'neighbours'.

1. Properties of the evolution of integer-valued stock vectors after one step

For a better understanding of the evolution towards a fixed point in an integer-valued model, in the first stage the behaviour after one step is examined for two initial stock vectors that are 'neighbours' in the sense that the corresponding coordinates differ at most with 1.

Considering a distance function $d : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^+$ and according to the procedure introduced in [4], for each $n \in C_k$ an associated integer-valued vector $\bar{n} \in C_k \cap \mathbb{N}^k$ can be determined: $\bar{n}$ is a vector of $C_k \cap \mathbb{N}^k$ 'nearest' to n for the distance function d. Therefore $\bar{n}$ is the result of the minimization problem:

$$\text{Min } d(w, n) \quad \text{with } w \text{ restricted to the set } C_k \cap \mathbb{N}^k.$$

Depending on the values of the coordinates of $n \in C_k$, there does exist exactly one associated integer-valued vector or there do exist more than one integer-valued vector that can be associated with n. In what follows the notation $\bar{n}$ refers to an integer-valued vector that can be associated with n.

Unless differently mentioned, the properties that are formulated in this paper hold for any integer-valued vector $\bar{n} \in C_k \cap \mathbb{N}^k$ that is a solution of $\mathrm{Min}_{w \in C_k \cap \mathbb{N}^k} d(w, n) = d(\bar{n}, n)$.

In practice however one can have good reasons to prefer a particular associated integer-valued vector above another one. For example in manpower planning where the components of an integer-valued stock vector refer to the number of employees in the different personnel categories, among the associated integer-valued vectors this one can be preferred that results in a minimum total personnel cost.

For initial integer-valued stock vectors v and $u \in N_1(v) = \{w \in C_k \cap \mathbb{N}^k | \|w - v\|_\infty = 1\}$ the vector $u \cdot A$ and the associated integer-valued vector $\overline{u \cdot A}$ are compared with the stock vector $\overline{v \cdot A}$. In this context it is worth to point out that a very small discrepancy between computed vectors $v \cdot A$ and $u \cdot A$, for some stock vectors v and u, does not guarantee that the associated integer-valued vectors $\overline{v \cdot A}$ and $\overline{u \cdot A}$ are equal.

Let us introduce the notation $N_{\leqslant p}(n) = \{w \in C_k \cap \mathbb{N}^k | \|w - n\|_\infty \leqslant p\}$ for the set of vectors w for which the coordinates differ at most with p from the corresponding coordinates of n.

And in an analogous way, let us denote $N_{\geqslant p}(n) = \{w \in C_k \cap \mathbb{N}^k | \|w - n\|_\infty \geqslant p\}$ and $N_p(n) = \{w \in C_k \cap \mathbb{N}^k | \|w - n\|_\infty = p\}$.

According to previous results in [4], with $n \in C_k$ the integer-valued vector $\bar{n}$ can be associated if:

$$\begin{aligned} n \in C(\bar{n}) &= \left\{ r \in C_k | d(\bar{n}, r) = \min_{w \in C_k \cap \mathbb{N}^k} d(w, r) \right\} \\ &= \left\{ r \in C_k | r_i - r_j \leqslant \bar{n}_i - \bar{n}_j + 1 \text{ for } i, j = 1, \ldots, k \right\} \end{aligned}$$

for a distance corresponding with a l-norm, i.e. $d_l(w, r) = \sqrt[l]{\sum_i |w_i - r_i|^l}$, $l \in \mathbb{N}_0$.

Furthermore it is pointed out in [5], that the set $C(\bar{n})$ is not depending on the considered l-norm and can be described as the convex hull

$$C(\bar{n}) = \mathrm{conv}\{\bar{n} + v(K_h) | K_h \subset K = \{1, \ldots, k\} \quad \text{with } \#K_h = h \text{ and } \; h = 1, \ldots, k-1\}$$

with for each number $h = 1, \ldots, k-1$ and for each subset $K_h \subset K = \{1, \ldots, k\}$ with $\#K_h = h$, the vector $v(K_h)$ defined as follows:

$$\begin{aligned} v_i(K_h) &= -1 + \tfrac{h}{k} && \text{for } i \in K_h, \\ v_i(K_h) &= \tfrac{h}{k} && \text{for } i \in K \setminus K_h. \end{aligned}$$

In order to be able to prove some properties of $\overline{u \cdot A}$ and $\overline{v \cdot A}$ for 'neighbours' $u, v \in C_k \cap \mathbb{N}^k$, the following lemmas are formulated.

Lemma 1.1. *For* $n \in C(\bar{n}) : \|n - \bar{n}\|_\infty \leqslant \frac{k-1}{k}$.

Proof. For $n \in C(\bar{n}) = conv\{\bar{n} + v(K_h) | K_h \subset K = \{1, \ldots, k\}$ with $\#K_h = h$ and $h = 1, \ldots, k-1\}$ holds that $|n_i - \bar{n}_i| \leqslant \max_{i, K_h} |v_i(K_h)| = \frac{k-1}{k}$ $\forall i$. Implying that $\|n - \bar{n}\|_\infty \leqslant \frac{k-1}{k}$. $\square$

Lemma 1.2. *For* $n \in C_k \cap \mathbb{N}^k : \left\{ r \in C_k | \|r - n\|_\infty \leqslant 1 + \frac{1}{k} \right\} \subset \bigcup_{w \in N_{\leqslant 1}(n)} C(w)$.

Proof. For a vector $r \in C_k \setminus \bigcup_{w \in N_{\leqslant 1}(n)} C(w)$ there does exist $s \in C_k \setminus N_{\leqslant 1}(n)$ such that $r \in C(s)$.

Since $s \in C_k \setminus N_{\leqslant 1}(n)$ and $s, n \in \mathbb{N}^k$: $\|s - n\|_\infty \geqslant 2$ and therefore

$$\begin{aligned}\|r - n\|_\infty &\geqslant \|s - n\|_\infty - \|r - s\|_\infty \\ &\geqslant 2 - \frac{k-1}{k} \quad \text{(according to Lemma 1.1, } r \in C(s)) \\ &= 1 + \frac{1}{k}.\end{aligned}$$

Therefore $\{r \in C_k \| \|r - n\|_\infty < 1 + \frac{1}{k}\} \subset \bigcup_{w \in N_{\leqslant 1}(n)} C(w)$.

And moreover since $\bigcup_{w \in N_{\leqslant 1}(n)} C(w)$ is a closed set:

$$\left\{r \in C_k \| \|r - n\|_\infty \leqslant 1 + \frac{1}{k}\right\} \subset \bigcup_{w \in N_{\leqslant 1}(n)} C(w) \qquad \square$$

Lemma 1.3. *For $r \in \bigcup_{w \in N_{\geqslant 2}(n)} C(w)$, $\exists l, m : r_l - r_m \geqslant n_l - n_m + 2$.*

Proof. If $r \in \bigcup_{w \in N_{\geqslant 2}(n)} C(w)$, then $\exists w \in N_{\geqslant 2}(n), \exists p \in \{1, \ldots, k\} : r \in C(w)$ and $|w_p - n_p| \geqslant 2$, i.e. $w_p = n_p + c_p$ with $|c_p| \geqslant 2$.

(a) If c_p is positive, then there does exist $q \in \{1, \ldots, k\} : w_q = n_q + c_q$ with $c_q \leqslant -1$ since $w, n \in C_k$ (and therefore $\sum_i w_i = \sum_i n_i$).

$$\begin{aligned}r \in C(w) \Rightarrow & r_q - r_p \leqslant w_q - w_p + 1 \\ \Rightarrow & r_q - r_p \leqslant (n_q + c_q) - (n_p + c_p) + 1 \\ \Rightarrow & r_q - r_p \leqslant n_q - n_p - 2 \\ \Rightarrow & r_p - r_q \geqslant n_p - n_q + 2,\end{aligned}$$

which proves the lemma for c_p positive.

(b) If c_p is negative, then $\exists q \in \{1, \ldots, k\} : w_q = n_q + c_q$ with $c_q \geqslant 1$. Consequently $r_q - r_p \geqslant n_q - n_p + 2$ which proves the lemma. $\square$

In [5], it is pointed out that there can be a substantial discrepancy between the evolution of $v \cdot A^{\mathrm{t}}$ on the one hand and the evolution of the stock vectors in the integer-valued model on the other hand.

To get a better understanding of the evolution of the subsequent integer-valued stock vectors depending on the starting stock vector, in what follows, for two initial stock vectors having coordinates differing at most with 1, the stock vectors after one step are examined. A characterisation concerning the stock vector $\overline{u \cdot A}$ after one step is given for a 'neighbour' $u \in N_1(v)$ of $v \in C_k \cap \mathbb{N}^k$.

In the following theorem different types of conditions on the $(k \times k)$ transition matrix A are formulated under which initial stock vectors having coordinates that differ at most 1, result after one step in stock vectors of which the coordinates still differ at most 1.

Let us introduce the notation A_l for the l-th column of the row-stochastic matrix $A = (a_{ij})$. And let us denote for A_l indices $i_1, \ldots, i_k$ satisfying $a_{i_1 l} \geqslant a_{i_2 l} \geqslant \cdots \geqslant a_{i_k l}$ and let us introduce the notations:

$$G_l = \left\{ i_1, i_2, \ldots, i_{\left[\frac{k}{2}\right]} \right\} \quad \text{and} \quad S_l = \left\{ i_{\left[\frac{k+1}{2}\right]+1}, \ldots, i_k \right\},$$

where the notation $[p]$ (for $p \in \mathbb{R}$) refers to the largest integer less than or equal to p.

Theorem 1.4. *For a transition matrix A satisfying one of the following conditions*:

(a) $\|A_l - A_m\|_1 < 1 \quad \forall l, m \in K = \{1, \ldots, k\}$ *or*

(b) $\sum_{i \in G_l} a_{il} - \sum_{i \in S_l} a_{il} \leqslant \frac{2}{k} \quad \forall l \in K$

holds that:

$$\overline{u \cdot A} \in N_{\leqslant 1}(\overline{v \cdot A}) \quad \forall v \in C_k \cap \mathbb{N}^k \; \forall u \in N_1(v).$$

Proof. For $v \in C_k \cap \mathbb{N}^k$ and $u \in N_1(v)$, $\exists K_+ \subset K = \{1, \ldots, k\}$ and $\exists K_- \subset K$ with $K_+ \cap K_- = \emptyset$ and $\#K_+ = \#K_-$ such that $u = v + \sum_{i \in K_+} e_i - \sum_{i \in K_-} e_i$, with $e_i \in \mathbb{N}^k$ the vector with all components equal to zero except the ith component that is equal to one.

Consequently

$$(u \cdot A)_l = (v \cdot A)_l + \sum_{i \in K_+} a_{il} - \sum_{i \in K_-} a_{il} \quad \forall l \in K. \tag{1.1}$$

Therefore under condition (a), for all $l, m \in K$ holds

$$\begin{aligned}
(u \cdot A)_l - (u \cdot A)_m &= (v \cdot A)_l - (v \cdot A)_m + \sum_{i \in K_+} (a_{il} - a_{im}) - \sum_{i \in K_-} (a_{il} - a_{im}) \\
&\leqslant (\overline{v \cdot A})_l - (\overline{v \cdot A})_m + 1 + \sum_{p \in K} |a_{pl} - a_{pm}| \quad \text{since } v \cdot A \in C(\overline{v \cdot A}) \\
&< (\overline{v \cdot A})_l - (\overline{v \cdot A})_m + 2 \quad \text{since } \|A_l - A_m\|_1 < 1.
\end{aligned}$$

Implying that $u \cdot A \in \bigcup_{w \in N_{\leqslant 1}(\overline{v \cdot A})} C(w)$ according to Lemma 1.3, which proves the theorem under condition (a).

Under condition (b), for all $l \in K$ holds

$$\begin{aligned}
&|(u \cdot A)_l - (\overline{v \cdot A})_l| \\
&\quad\leqslant |(u \cdot A)_l - (v \cdot A)_l| + |(v \cdot A)_l - (\overline{v \cdot A})_l| \\
&\quad\leqslant \left| \sum_{i \in K_+} a_{il} - \sum_{i \in K_-} a_{il} \right| + \frac{k-1}{k} \quad \text{according to expression(1.1) and Lemma 1.1} \\
&\quad\leqslant \sum_{i \in G_l} a_{il} - \sum_{i \in S_l} a_{il} + \frac{k-1}{k} \\
&\quad\leqslant 1 + \frac{1}{k} \quad \text{under condition (b).}
\end{aligned}$$

Implying that $u \cdot A \in \bigcup_{w \in N_{\leqslant 1}(\overline{v \cdot A})} C(w)$ according to Lemma 1.2, which proves the theorem under condition (b). $\square$

2. Properties of the evolution towards a limiting stock vector

In the previous part, for an initial vector $u \in C_k \cap \mathbb{N}^k$ properties of the integer-valued stock vector $\overline{u \cdot A}$ after one step are discussed. In what follows depending on the starting stock vector the limiting stock vector for the integer-valued model is characterised.

On the one hand, there do exist transition matrices A and initial stock vectors u for which there is no convergence towards a limiting stock vector for the integer-valued model.

For example, for $A = \begin{pmatrix} 0.05 & 0.9 & 0.05 \\ 0.05 & 0.05 & 0.9 \\ 0.9 & 0.05 & 0.05 \end{pmatrix}$ and $u = (2, 4, 4)$ the stock vector does not converge, since the evolution is cyclic: $(2, 4, 4) \to (4, 2, 4) \to (4, 4, 2) \to (2, 4, 4) \to \cdots$

On the other hand, if there is convergence, the limiting stock vector is necessarily a fixed point $f \in F$. The set F of fixed points is characterised in [5] as the set of integer-valued vectors in a well defined symmetric convex set with point of symmetry $n^* \in C_k$, the unique fixed point for the transition matrix A of the regular Markov chain.

For a transition matrix A, let us introduce the notation $L(A)$ for the set of integer-valued stock vectors for which there do exist subsequent integer-valued vectors that are converging for the integer-valued model with transition matrix A:

$$L(A) = \left\{ u(0) \in C_k \cap \mathbb{N}^k | \forall t \in \mathbb{N}, \exists \overline{u(t+1)}: \overline{u(t)} \cdot A \in C(\overline{u(t+1)}) \text{ with } \lim_{t \to +\infty} \overline{u(t)} \ \exists \right\}.$$

And let us denote $L_f(A)$ the set of integer-valued stock vectors for which there do exist subsequent integer-valued vectors evolving towards $f \in F$:

$$L_f(A) = \left\{ u(0) \in L(A) | \ \forall t \in \mathbb{N}, \exists \overline{u(t+1)}: \overline{u(t)} \cdot A \in C(\overline{u(t+1)}) \text{ with } \lim_{t \to +\infty} \overline{u(t)} = f \right\}.$$

It is obvious that for a fixed point $f \in F$ holds that $f \in L_f(A)$. In the following theorem a further characterisation of $L_f(A)$ is proved for an internal fixed point f of F, i.e. a fixed point for which the 'neighbours' $N_1(f) = \{w \in C_k \cap \mathbb{N}^k | \|w - f\|_\infty = 1\}$ are also fixed points.

Theorem 2.1. *For $f \in F$ such that $N_1(f) \subset F : \forall g \in L(A) \setminus \{f\}, \exists h \neq f : g \in L_h(A)$.*

Proof. In case $N_1(f) \subset F$, for $g \in N_1(f) \setminus \{f\}$ holds that $g \in L_h(A)$ with $h = g \neq f$.

Which proves the theorem. $\square$

Corollary 2.2. *In case the unique fixed point $n^* \in C_k$ of the regular Markov chain with transition matrix A is integer-valued with $N_1(n^*) \subset F$, then for the integer-valued model all initial stock vectors $n \in L(A)$ different from n^* do converge (for particular subsequent integer-valued stock vectors) towards a limiting stock vector that is not equal to n^*.*

Remark that the result of Corollary 2.2 for integer-valued models is in contrast with the situation for regular Markov chains, since for regular Markov chains the Fixed-Point Theorem implies that for any starting stock vector the limiting stock vector is equal to n^*.

In what follows, under certain conditions on $v \in C_k \setminus F$ the evolution and the limiting stock vector for the integer-valued model with transition matrix A is characterized.

Lemma 2.3. *For* $v \in C_k \setminus F$:

$$\begin{array}{ll} \text{If} & (v\cdot(A-I))_i-(v\cdot(A-I))_j=\max\limits_{l,m}[(v\cdot(A-I))_l-(v\cdot(A-I))_m]\leqslant 2 \text{ and} \\ & (v\cdot(A-I))_l-(v\cdot(A-I))_m\leqslant 1 \quad \forall l,m\notin\{i,j\} \\ \text{then} & v\cdot A\in C(v+e_i-e_j). \end{array}$$

Proof. The condition for $v \cdot A$ to be an element of $C(v+e_i-e_j)$ can be reformulated as

$$\begin{aligned} &(v\cdot A)_l-(v\cdot A)_m\leqslant(v+e_i-e_j)_l-(v+e_i-e_j)_m+1 \quad \forall l,m\in\{1,\dots,k\} \\ &\Leftrightarrow (v\cdot(A-I))_l-(v\cdot(A-I))_m\leqslant\delta_{il}-\delta_{jl}-\delta_{im}+\delta_{jm}+1 \quad \forall l,m\in\{1,\dots,k\}. \end{aligned} \tag{2.1}$$

For $v \in C_k \setminus F$ holds that $v \cdot A \notin C(v)$, implying that

$$\max_{l,m}[(v\cdot(A-I))_l-(v\cdot(A-I))_m]>1. \tag{2.2}$$

Furthermore, since $(v\cdot(A-I))_i-(v\cdot(A-I))_j=\max_{l,m}[(v\cdot(A-I))_l-(v\cdot(A-I))_m]$:

$$(v\cdot(A-I))_i=\max_l\,(v\cdot(A-I))_l \quad \text{and} \quad (v\cdot(A-I))_j=\min_m\,(v\cdot(A-I))_m. \tag{2.3}$$

Since

$$\begin{array}{ll} (v\cdot(A-I))_j-(v\cdot(A-I))_i<-1 & \text{according to (2.2)}, \\ (v\cdot(A-I))_l-(v\cdot(A-I))_i\leqslant 0 & \forall l\in\{1,\dots,k\},\ \text{according to (2.3)}, \\ (v\cdot(A-I))_j-(v\cdot(A-I))_m\leqslant 0 & \forall m\in\{1,\dots,k\},\ \text{according to (2.3)} \end{array}$$

and since under the conditions stated:

$$\begin{aligned} &(v\cdot(A-I))_l-(v\cdot(A-I))_m\leqslant 2 \text{ in case } l=i \text{ and/or } m=j \\ &\qquad\qquad\qquad \text{because } \max_{l,m}[(v\cdot(A-I))_l-(v\cdot(A-I))_m]\leqslant 2 \\ &(v\cdot(A-I))_l-(v\cdot(A-I))_m\leqslant 1 \text{ in case } l,m\notin\{i,j\} \end{aligned}$$

the inequalities (2.1) are fulfilled and therefore $v\cdot A\in C(v+e_i-e_j)$. $\square$

Theorem 2.4. *For* $v \in C_k \setminus F$:

$$\begin{array}{ll} \text{If} & (v\cdot(A-I))_i-(v\cdot(A-I))_j=\max\limits_{l,m}\left[(v\cdot(A-I))_l-(v\cdot(A-I))_m\right]\leqslant 2 \text{ and} \\ & (v\cdot(A-I))_l-(v\cdot(A-I))_m\leqslant 1 \quad \forall l,m\notin\{i,j\} \text{ and} \\ & v+e_i-e_j\in F \\ \text{then} & v\in L_{v+e_i-e_j}(A) \end{array}$$

Proof. Under the formulated conditions, according to Lemma 2.3: $v\cdot A\in C(v+e_i-e_j)$ and therefore $v\in L_{v+e_i-e_j}(A)$ since $v+e_i-e_j\in F$. $\square$

Theorem 2.5. *For a transition matrix A satisfying one of the following conditions*:

(a) $\|A_l-A_m\|_1<1 \quad \forall l,m\in K$ *or*

(b) $\sum_{i\in G_l}a_{il}-\sum_{i\in S_l}a_{il}\leqslant\frac{2}{k} \quad \forall l\in K$

and for $u,v\in L(A)$ *with* $\|u-v\|_\infty\leqslant 1$

holds that $u \in L_f$ and $v \in L_g$ for f and g satisfying $\|f-g\|_\infty \leqslant 1$.

Theorem 2.5 is a direct consequence of Theorem 1.4: under the stated conditions on the transition matrix A initial stock vectors having coordinates that differ at most with one, result after one step in integer-valued stock vectors still having coordinates that differ at most with one; and since this property is applicable after each step, the same conclusions remain at each stage in the evolution as well as for the limiting stock vectors.

3. Illustrations

For the number of states $k > 3$ and the transition matrix $A(k)$ defined as follows:

$$\begin{array}{ll} a_{ii}(k) = \frac{2}{k} & \text{for } i = 1, \ldots, k, \\ a_{ij}(k) = \frac{k-2}{k(k-1)} & \text{for } j \neq i \end{array}$$

holds that

$$\|A_l - A_m\|_1 = \frac{2}{k-1} < 1 \quad \forall l, m \in K.$$

Therefore, the condition (a) formulated in Theorem 1.4 and Theorem 2.5 is fulfilled for the matrix $A(k)$. Consequently, in the integer-valued model with transition matrix $A(k)$ two arbitrary starting vectors that are 'neighbours', do have a trajectory of evolution that is similar since the subsequent integer-valued vectors are 'neighbours' at each stage.

On the other hand, for the number of states $k > 3$ and the transition matrix $A(k)$ with elements:

$$\begin{array}{l} a_{i,1}(k) = 1 - \frac{1}{k} \text{ for } i = 1, \ldots, \left[\frac{k}{2}\right], \quad a_{i,2}(k) = 0 \text{ for } i = 1, \ldots, \left[\frac{k}{2}\right], \\ a_{i,1}(k) = 0 \text{ for } i = \left[\frac{k}{2}\right] + 1, \ldots, k, \quad a_{i,2}(k) = 1 - \frac{1}{k} \text{ for } i = \left[\frac{k}{2}\right] + 1, \ldots, k, \\ a_{ij}(k) = \frac{1}{k(k-2)} \quad \text{for } i = 1, \ldots, k \text{ and } j = 3, \ldots, k \end{array}$$

holds that

$$\|A_1 - A_2\|_1 = k - 1 > 1 \;\text{ and }\; \sum_{i \in G_1} a_{i1} - \sum_{i \in S_1} a_{i1} = \left[\frac{k}{2}\right] \cdot \left(1 - \frac{1}{k}\right) \geqslant \frac{(k-1)^2}{2k} > \frac{2}{k}.$$

Therefore, condition (a) as well as condition (b), formulated in Theorem 1.4 and Theorem 2.5, is not fulfilled.

For example for the 'neighbours' v and $u = v + \sum_{i=1}^{[\frac{k}{2}]} e_i - \sum_{i=[\frac{k}{2}]+1}^{2[\frac{k}{2}]} e_i$ the situation is that:

$$\|u \cdot A(k) - v \cdot A(k)\|_\infty = \left[\frac{k}{2}\right] \cdot \left(1 - \frac{1}{k}\right) \geqslant \frac{(k-1)^2}{2k}.$$

Consequently for greater values of k, the 'neighbours' u and v result after one step in $\overline{u \cdot A(k)}$ and $\overline{v \cdot A(k)}$ that are not similar and do have trajectories of evolution that are not similar.

4. Conclusions and further research questions

For the integer-valued model based on a regular Markov chain, there do exist transition matrices and initial stock vectors for which the evolution is cyclic (illustration in Section 2). Under such circumstances, there is no convergence towards a limiting stock vector in the integer-valued model.

In Theorem 1.4 and Theorem 2.5, conditions on the transition matrix are formulated under which two initial stock vectors that are 'neighbours', do have similar trajectories of evolution with at each intermediate step integer-valued stock vectors that are 'neighbours'. Under these conditions on the transition matrix, in case of convergence, the limiting stock vectors are still 'neighbours'.

For a transition matrix A, having for the integer-valued model a set F of fixed points containing more than one element, the challenge for further research is to find a description of the tilling of C_k for which all starting stock vectors in a same till do converge to the same limiting stock vector for the integer-valued model.

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